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Ultrarelativistic Bose–Einstein gas on Lorentz symmetry violation

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ABSTRACT

In this paper, we study the effects of Lorentz Symmetry Breaking on the thermodynamic properties of ideal gases. Inspired by the dispersion relation coming from the Carroll–Field–Jackiw model for Electrodynamics with Lorentz and CPT violation term, we compute the thermodynamics quantities for a Boltzmann, Fermi–Dirac and Bose–Einstein distributions. Two regimes are analyzed: the large and the small Lorentz violation. In the first case, we show that the topological mass induced by the Chern–Simons term behaves as a chemical potential. For Bose–Einstein gases, a condensation in both regimes can be found.

0. Introduction

Since the advent of the String Theories as a candidate for a Unified Theory, Lorentz- and CPT- Violations are expected at Planck Scale [1], and a background anisotropy on the space–time must correct the physics at a low energy scale. For instance, at a Standard Model scale, the Standard Model Extension SME [2] was proposed as a possible extension of the minimal Standard Model of the fundamental interactions. Even with the expectation of the Lorentz- and CPT violation, these effects are very small, and the SME has also been used as a framework to get stringent bounds on the Lorentz-symmetry violating (LV) coefficients [3,4]. More recent results on these bounds can be found in Ref. [5]. In this framework, there are a large number of results in the literature that investigate these effects in different situations, like system involving photons [6,7], radiative corrections [8], fermions [9], neutrinos [10], topological defects [11], topological phases [12], cosmic rays [13], supersymmetry [14], particle decays [15], and other relevant aspects [16,17].

This violation can be implemented in the fermion sector, for example, by $a_\mu \overline{\psi}\gamma^\mu\psi$, $b_\mu\overline{\psi}\gamma_5\gamma^\mu\psi$, leading to a modified Dirac theory [18]. It also has consequences in a very low energy regime, like in atomic physics, condensed matter and so on, taking into account the non relativistic limit, in order to obtain experimental bounds on the Lorentz symmetry braking (LSB) parameters and other effects, such as the generation of an anomalous magnetic moment by a non-minimal coupling covariant derivative [19,20].

In the gauge sector, it has been introduced in the pioneering work by Carrol et al. [21], modifying the Electromagnetic Maxwell Lagrangian by means of a Chern–Simons type extra term ($\propto\ \epsilon^{\mu\alpha\beta\gamma}(k_{AF})_\mu F_{\alpha\beta}A_\gamma$). This modifies the dispersion relation of the photon introducing a topological mass and preserving the gauge invariance.

Even though LSB is a theoretical implication of more fundamental theories, the search for a deviation on the photon dispersion relation has important consequences in cosmic physics, like Gamma Ray Bursts, near stars with strong magnetic field and vacuum birefringence. This can justify the study of the statistical behavior of the photon gas under a non-linear dispersion relation. As this violation is expected to be significantly observed only at ultra-high energy, this addresses to the question of its statistical behavior in a ultrarelativistic photon gas, not only to get more stringent bounds on the LSB parameters but also an interesting phenomenon that could be related by this breaking.

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Another interesting point is related to the light bosons [22], which are described by a scalar field that, minimally coupled to gravity, could be a candidate for Dark Matter. Its mass is constrained to the order 10^{-22} eV. In this reference, the authors argue that the neutrino radiation behavior is linked to a ultrarelativistic transition. This scalar field falls in the classification of Hot Dark Matter (HDM), in the sense that it behaves as radiation at its decoupling epoch. This is related to a Bose–Einstein condensation in ultrarelativistic gas. These facts motivate the study of a Bose–Einstein gas with a non-linear dispersion relation at the ultrarelativistic level.

In this paper, we study the Statistical Mechanics of an ideal gas embedded in a LSB background, starting from a massive Chern–Simons dispersion relation. Two regimes are computed: the Large Lorentz Violation (LLV), when the background is stronger than the momentum of the particle and the Small Lorentz Violation (SLV) when the background is weaker than that momentum. We will show that in this limit our results are not the same as those obtained when started by the SME Lagrangian where a non-relativistic Hamiltonian is computed for the free Bose–Einstein, Fermi Dirac and Boltzmann gas [23], since the regimes studied here are not the same. In our case we get our results using the background as the reference parameter on the expansion not a mass. We shall see, for the Bose gas, that LSB background can induce a phase transition for the high energy gas.

This paper is organized as follows. In the Section 1 we study the Boltzmann gas at the LLV and SLV limit, where we recognize the role of the LSB parameter as a chemical potential as noted in Ref. [24]. In the Section 2, we study the Fermi–Dirac statistics and calculate the corrections on the thermodynamic quantities coming from the LSB background. The Section 3 is devoted to the Bose–Einstein distribution where we study how the background affects the thermodynamic quantities as well as the critical conditions for the formation of a Bose–Einstein condensate induced by the vector background that gives rise to the possibility of obtaining a state depending on this value. In the last section we present our final comments.

1. Boltzmann's gas

As a first step, we analyze the Statistical Mechanics for an ideal Boltzmann gas, by computing the LLV and the SLV limit. The gas is described by the dispersion relation raised from the Carroll–Field–Jackiw model [21], which is given by, $p^4+k_{AF}^2p^2-(k_{AF}\cdot p)^2=0$, where k_{AF} is a four-vector background field. This background violates CPT and Lorentz symmetries in the sense of particle frame transformations. Obeying causality conditions we take the background vector as being space-like, $k_{AF}^{\mu}=(0,\vec{k}_{AF})$ as described in Ref. [25] and $k_{AF}=|\vec{k}_{AF}|$. In this regime, ϵ is given by

$$\epsilon=\sqrt{\frac{2\vec{p}^2+\vec{k}_{AF}^2}{2}\pm\frac{1}{2}\sqrt{\vec{k}_{AF}^4+4(\vec{k}_{AF}\cdot\vec{p})^2}}. \tag{1}$$

In statistical mechanics the partition function for a system of N particles is given by

$$\Xi\,(T,V,\mu)=\sum_{N=0}^{\infty}z^NZ\,(T,V,N) \tag{2}$$

where $Z\,(T,V,N)=\frac{1}{N!}Z^N\,(T,V,1)$ and $Z\,(T,V,1)=\frac{1}{h^3}\int \mathrm{e}^{-\beta E(p,q)}\mathrm{d}p\mathrm{d}q$ with $\beta=\frac{1}{kT}$ and $z=\mathrm{e}^{\beta\mu}$. In order to split the two regimes that we are interested in, we analyze the limit cases, when $|k_{AF}|\gg|\vec{p}|$, that is the LLV regime of the (1)

$$\epsilon_{\pm}=k_{AF}+\frac{1}{2}\frac{p^2}{k_{AF}}(1\pm\cos^2\theta) \tag{3}$$

and the SLV case, $k_{AF}\ll|\vec{p}|$, the first order on k_{AF} of (1) assumes

$$\epsilon=|\vec{p}|+\frac{1}{2}|\cos\theta|k_{AF}. \tag{4}$$

1.1. The large Lorentz violation limit: $k_{AF}\gg|\vec{p}|$

In order to study the Statistical Mechanics of the classical gas in the LLV limit, we insert the dispersion relation (3) in the partition function (2). Here, we shall not consider the case with negative energy, by means that this situation violates the causality conditions, [25]. Then,

$$Z\,(T,V,1)=\frac{2\pi V\mathrm{e}^{-\beta k_{AF}}}{h^3}\int_0^{\pi}\int_0^{\infty}\sin\theta\,\mathrm{e}^{-\beta\left[\frac{1}{2}\frac{p^2}{k_{AF}}(1+\cos^2\theta)\right]}p^2\mathrm{d}p\mathrm{d}\theta, \tag{5}$$

after a straightforward integration, we obtain the Z-function $Z\,(T,V,1)=\frac{1}{4}\left(\frac{2k_{AF}}{2\pi\beta}\right)^{3/2}V\mathrm{e}^{-\beta k_{AF}}$.

Thus, the partition function is

$$\Xi\left(T, V, \mu\right) = \sum_{N=0}^{\infty} \frac{1}{N!4^N} \left(\frac{2k_{AF}}{2\pi\beta}\right)^{\frac{3}{2}N} V^N \mathrm{e}^{\beta(\mu-k_{AF})N}. \tag{6}$$

To compute the number of particles $N = -\frac{\partial\phi}{\partial\mu}$, pressure $P = -\frac{\partial\phi}{\partial V}$ and the entropy $S = -\frac{\partial\phi}{\partial T}$, we calculate the grand-potential function

$$\phi = -kT \ln \Xi = -kT\mathrm{e}^{\beta(\mu-k_{AF})} \frac{1}{4} \left(\frac{2k_{AF}}{2\pi\beta}\right)^{3/2} V, \tag{7}$$

that results, respectively in

$$\begin{aligned} N &= \mathrm{e}^{\beta(\mu-k_{AF})} \frac{1}{4} \left(\frac{2k_{AF}}{2\pi\beta}\right)^{3/2} V, \\ P &= \frac{kTN}{V}, \\ S &= kN \left[\frac{5}{2} - \left(\mu - k_{AF}\right)\beta\right]. \end{aligned} \tag{8}$$

The energy can be computed by the relation, $U = TS - \mathrm{PV} + \mu N$ which results in $U = \frac{3}{2}NkT + k_{AF}N$, and the state equation $U = \left(\frac{3}{2} + \beta k_{AF}\right)\mathrm{PV}$. The chemical potential can be calculated explicitly in this case as

$$\mu = -kT \left[\ln\left(\frac{N_Q}{N_C}\right) - \left(\frac{k_{AF}}{kT} + \ln 4 - 1\right)\right], \tag{9}$$

where we call the quantities $N_Q = \left(\frac{2k_{AF}}{2\pi\beta}\right)^{\frac{3}{2}}$ as the quantum concentration and $N_C = \frac{N}{V}$ the standard classical concentration, or density of particles. Note that the chemical potential is affected by the background field k_{AF} as was reported by Colladay et al. [23]. This change is due to the behavior of the field as an effective mass for the gas.

In the thermodynamic limit, and when $kT \gg k_{AF}$, we obtain

$$C_V = \frac{\partial U}{\partial T} = \frac{3}{2}Nk. \tag{10}$$

This result looks like a non-relativistic ideal gas in the Boltzmann's distribution.

1.2. The small Lorentz violation limit $k_{AF} \ll |\vec{p}|$

With a similar procedure as the one adopted in the LLV limit, we find for the Z-function $Z\left(T, V, 1\right) = \frac{8V}{(2\pi)^2\beta^4 k_{AF}} \mathrm{e}^{-\frac{\beta k_{AF}}{2}} (\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1)$, that the partition function and the grand potential become

$$\begin{aligned} \Xi\left(T, V, \mu\right) &= \exp\left[\mathrm{e}^{\beta\mu} \frac{8V}{(2\pi)^2\beta^4 k_{AF}} \mathrm{e}^{-\frac{\beta k_{AF}}{2}} \left(\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1\right)\right] \\ \phi &= -kT \left[\mathrm{e}^{\beta\mu} \frac{8V}{(2\pi)^2\beta^4 k_{AF}} \mathrm{e}^{-\frac{\beta k_{AF}}{2}} \left(\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1\right)\right]. \end{aligned} \tag{11}$$

Then, the particle number is

$$N = \mathrm{e}^{\beta\mu} \frac{8V}{(2\pi)^2\beta^4 k_{AF}} \mathrm{e}^{-\frac{\beta k_{AF}}{2}} \left(\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1\right).$$

The entropy, energy and the specific heat C_V with the background are respectively

$$S = kN \left[5 - \mu\beta - \frac{k_{AF}}{2}\beta - \frac{k_{AF}}{2}\beta \left(\frac{\mathrm{e}^{\frac{\beta k_{AF}}{2}}}{\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1}\right)\right] \tag{12}$$

$$U = \left[4 - \left(\frac{\frac{\beta k_{AF}}{2}}{\mathrm{e}^{\frac{\beta k_{AF}}{2}} - 1}\right)\right] \mathrm{PV} \tag{13}$$

$$C_V = 4kN - \frac{\mathrm{e}^{\frac{k_{AF}}{2kT}} N k_{AF}^2}{4\left(\mathrm{e}^{\frac{k_{AF}}{2kT}} - 1\right) kT^2}. \tag{14}$$

and we have the same state equation $\mathrm{PV} = NkT$.

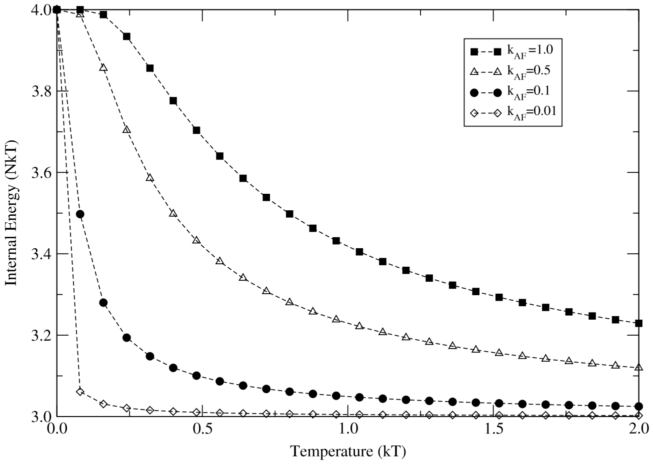

Fig. 1. Internal energy U as a function of kT. We used values $k_{AF} = 1.0, 0.5, 0.1, 0.01$. When $k_{AF} \to 0$ all curves will degenerate to $U = 3NkT$ for any temperature.

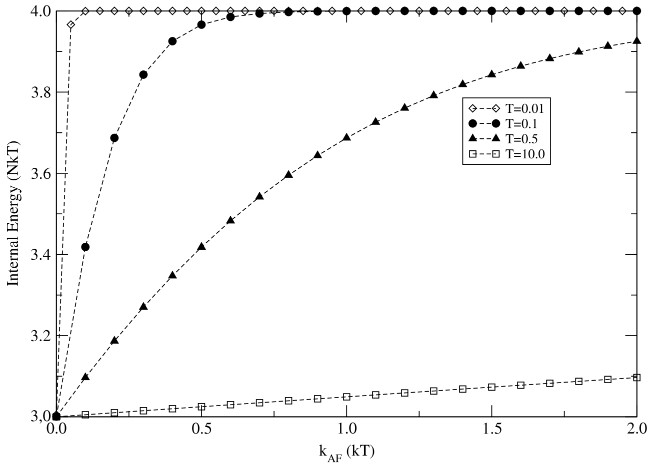

Fig. 2. Internal energy U as a function of k_{AF}. We used values $T = 0.01, 0.1, 0.5, 10.0$. When $k_{AF} \to 0$ all curves converge to $U = 3NkT$ for any temperature.

The chemical potential is now given by

$$\mu = kT\left[\ln\left(\frac{\pi^2\beta^4 k_{AF} N_C}{2}\right) - \left(\frac{k_{AF}}{kT} + \ln\left(1 - e^{-\frac{\beta k_{AF}}{2}}\right) - 1\right)\right]. \tag{15}$$

We can observe how the background parameter k_{AF} modifies the statistics. In the limit where the background does not exist, we recover the standard result $U = 3NkT$ and $C_V = 3Nk$. The behavior of the Boltzmann gas in the presence of background field is illustrated in the figures that follow. In Fig. 1, we can observe the behavior of the internal energy with the temperature and a fixed parameter k_{AF}. For large temperatures all the curves tend asymptotically to the value $U = 3NkT$ and collapse to this value when the parameter k_{AF} goes to zero. The same behavior can be seen in Fig. 2, where U is a function of k_{AF}. Note here that the field k_{AF} clearly introduces an effective mass in the system so that the internal energy increases. The behavior of the specific heat is shown in Fig. 3 and again we found the change introduced by the background field. When k_{AF} goes to zero at $T = 0$ the value of $C_V = 3Nk$, the same expected by Boltzmann's statistics for relativistic particles.

2. Fermi–Dirac statistics

We now begin to study the effects of the LSB background on the quantum gases. To this end, we should compute the Fermi–Dirac and Bose–Einstein distribution and in this paper we are interested in both limits, LLV and SLV regimes. In the Fermi–Dirac statistics, the grand potential is given by

$$\phi = -kT\,\frac{2\pi V}{h^3}\frac{\beta}{3}\int_0^{\pi}\int_0^{\infty}\sin\theta\, p^3\,\frac{ze^{-\beta\epsilon}}{1+ze^{-\beta\epsilon}}\left(\frac{d\epsilon}{dp}\right)dp\,d\theta. \tag{16}$$

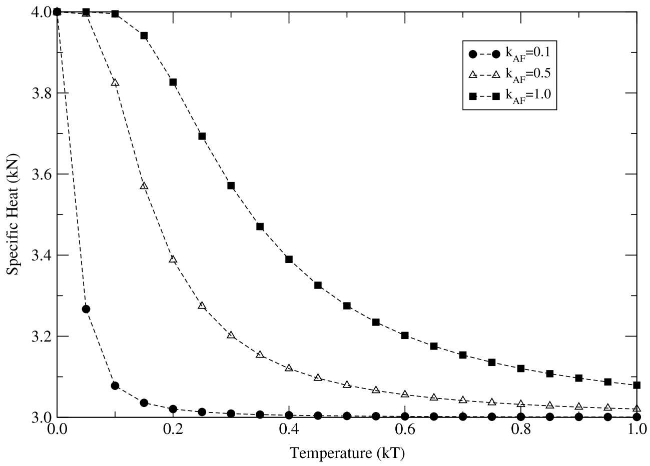

Fig. 3. Specific Heat C_V as a function of T. We used values $k_{AF} = 1.0, 0.5, 0.1$. When $k_{AF} \to 0$ all curves converge to $C_V = 3kN$ for any temperature.

The LLV regime, i.e. $k_{AF} \gg |\vec{p}|$, $\epsilon = k_{AF} + \frac{1}{2}\frac{p^2}{k_{AF}}(1 + \cos^2\theta)$ and $\frac{d\epsilon}{dp} = \frac{p}{k_{AF}}(1 + \cos^2\theta)$, which yields the grand partition function

$$\phi = -kT\,\frac{V}{\lambda^3}\frac{\sqrt{2}}{2}f_{5/2}(ze^{-\beta k_{AF}}). \tag{17}$$

We should note that the thermal wavelength of the Fermi gas λ, depends on the LSB parameter

$$\lambda = \left(\frac{h^2}{2\pi k_{AF} kT}\right)^{1/2} \tag{18}$$

and $f_n(\chi)$ is the complete Fermi–Dirac integral defined by

$$f_n(\chi) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{\xi^{n-1}}{\chi^{-1}e^{\xi} + 1}d\xi. \tag{19}$$

Some important thermodynamic quantities such as particle number, pressure and density of energy are defined below:

$$N = \frac{V}{\lambda^3}\frac{\sqrt{2}}{2}f_{3/2}(ze^{-\beta k_{AF}})$$
$$P = \frac{1}{\lambda^3}(kT)\frac{\sqrt{2}}{2}f_{5/2}(ze^{-\beta k_{AF}})$$
$$U = \frac{1}{\lambda^3}\frac{\sqrt{2}}{2}\left[k_{AF}f_{3/2}(ze^{-\beta k_{AF}}) + \frac{3}{2}(kT)f_{5/2}(ze^{-\beta k_{AF}})\right]V. \tag{20}$$

The equation of state for a Fermi gas is

$$\mathrm{PV} = NkT\frac{f_{5/2}(ze^{-\beta k_{AF}})}{f_{3/2}(ze^{-\beta k_{AF}})}.$$

For high temperatures, the Fermi gas behaves like a Boltzmann gas; then, the complete Fermi–Dirac function is $f_n(\chi) \approx \chi$, or $\mathrm{PV} = NkT$. These results are the same obtained by Colladay [23].

The SLV regime, i.e. $k_{AF} \ll |\vec{p}|$ and the energy $\epsilon = |\vec{p}| + \frac{1}{2}|\cos\theta|k_{AF}$, $\frac{d\epsilon}{dp} = 1$. The grand partition function can be written as

$$\phi = -kT\,\frac{8\pi V}{h^3}\frac{1}{\beta^3}I_5(z, \chi) \tag{21}$$

where the functions $I_n(z, \chi)$ are

$$I_n(z, \chi) = \frac{1}{\ln\chi}\left(f_n(z) - f_n(z\chi)\right). \tag{22}$$

In the limit when $k_{AF} \to 0$ the grand potential becomes

$$\phi = -kT\,\frac{8\pi V}{h^3}\frac{1}{\beta^3}f_4(z). \tag{23}$$

The pressure may be written like

$$P = \frac{8\pi}{h^3}(kT)^4 I_5(z, \chi). \tag{24}$$

For $k_{AF} \to 0$ and $kT \gg \mu$, we have

$$P = \frac{8\pi}{h^3}(kT)^4 \frac{\pi^4}{90}\frac{7}{8}.$$

We see that the pressure for Fermi–Dirac gas differs by the factor $\frac{7}{8}$ when compared to Bose gas in same condition $k_{AF} \to 0$ and $z \to 1$ as usual.

The internal energy U is

$$U = \frac{8\pi V}{h^3}(kT)^4(4I_5(z, \chi) - f_4(z\chi)),$$

when $k_{AF} \to 0$ and $z \to 1$

$$U = \frac{8\pi V}{h^3}(kT)^4 \frac{7\pi^4}{240}. \tag{25}$$

3. Bose–Einstein statistics

Colladay and McDonald [23] have studied the effects of the LSB background in a general non-relativistic statistics. The general effect of the background in this kind of situation is to redefine the thermodynamics quantities, unlike obtaining a new effect, since the background is expected to be small in the low energy scale. In this framework, these systems can be used to get experimental bounds on the LSB parameters. In a Bose gas, they have been analyzed in two very different situation. In a first publication, it has been shown how the background can modify the standard thermodynamic results as the critical temperature for the homogeneous non-relativistic Bose–Einstein Condensates (BEC) phase transition, starting from the non-relativistic Hamiltonian with LSB non-relativistic contribution. In a second publication it has been proposed to use the BEC trapped as a probe to LSB parameters.

Here, we start off from the Carroll–Field–Jackiw dispersion relation (1), and compute both limits. The LLV regime, in the same way, was computed for the Boltzmann gas (3). Our goal here, is the study of the ultrarelativistic statistics for free Bose gas under the same background. We will see that under this regime, a phase transition can be induced by the vector background.

The grand canonical partition function can be calculated by evaluating the number of states in the one-particle phase space Σ, where in spherical coordinates is $\Sigma = \frac{2\pi V}{h^3}\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^\infty p^2 \mathrm{d}p$, the one-particle density of states is given by

$$g(\epsilon) = \frac{\mathrm{d}\Sigma}{\mathrm{d}\epsilon} = \left[\frac{2\pi V}{h^3}\int_0^\pi \sin\theta \int_0^\infty p^2 \mathrm{d}p\right]\frac{\mathrm{d}p}{\mathrm{d}\epsilon}. \tag{26}$$

The grand canonical partition function, Ξ, for the Bose statistics is given by

$$\ln \Xi(T, V, z) = -\sum_k^\infty \ln(1 - z\mathrm{e}^{-\beta\epsilon_k}). \tag{27}$$

By computing the thermodynamic limit, $\ln \Xi = -\int_0^\infty g(\epsilon)\ln(1 - z\mathrm{e}^{-\beta\epsilon})\mathrm{d}\epsilon - \ln(1-z)$, then grand potential is

$$\phi = -kT\,\frac{2\pi V}{h^3}\frac{\beta}{3}\int_0^\pi \int_0^\infty \sin\theta\, p^3 \frac{z\mathrm{e}^{-\beta\epsilon}}{1 - z\mathrm{e}^{-\beta\epsilon}}\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}p}\right)\mathrm{d}p\,\mathrm{d}\theta - \ln(1-z) \tag{28}$$

where $z = \mathrm{e}^{\beta\mu}$ is the fugacity.

3.1. *The large Lorentz violation limit* $k_{AF} \gg |\vec{p}|$

The dispersion relation in the LLV limit is given by (3). To study the effects of LSB in this case we put this dispersion relation in (28). The grand potential function can be written as

$$\phi = -kT\,\frac{2\pi V}{h^3}\frac{\beta}{3}\int_0^\pi \int_0^\infty \sin\theta\, p^3 \frac{z\mathrm{e}^{-\beta\left[k_{AF} + \frac{1}{2}\frac{p^2}{k_{AF}}(1+\cos^2\theta)\right]}}{1 - z\mathrm{e}^{-\beta\left[k_{AF} + \frac{1}{2}\frac{p^2}{k_{AF}}(1+\cos^2\theta)\right]}}\frac{p}{k_{AF}}(1+\cos^2\theta)\mathrm{d}p\,\mathrm{d}\theta, \tag{29}$$

after evaluating the integral, we obtain

$$\phi = -kT\,\frac{V}{\lambda^3}\frac{\sqrt{2}}{2}g_{5/2}(e^{\beta(\mu-k_{AF})}) \tag{30}$$

where $g_n(z)$, is defined by $g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{z^{-1}e^x-1}$, $0 \le z \le 1$ and $\lambda = \left(\frac{h^2}{2\pi k_{AF} kT}\right)^{1/2}$, is the thermal wavelength.

Computing the thermodynamical quantities, we obtain

$$P = \frac{1}{\lambda^3}(kT)\frac{\sqrt{2}}{2}g_{5/2}(e^{\beta(\mu-k_{AF})}),$$

$$N = \frac{V}{\lambda^3}\frac{\sqrt{2}}{2}g_{3/2}(e^{\beta(\mu-k_{AF})}) + N_0. \tag{31}$$

The total number N is then $N = N_1 + N_0$ where N_0 is the number of particles in the ground state and $N_1 = \frac{V}{\lambda^3}\frac{\sqrt{2}}{2}$ $g_{3/2}(e^{\beta(\mu-k_{AF})})$ is the number of particles in excited states.

The critical temperature T_c when the Bose–Einstein condensation (BEC) occurs is obtained when $g_{3/2}(x)$ has a maximum, or $\mu = k_{AF}$.

$$T_c = \left(\frac{N}{V}\right)^{2/3}\frac{h^2}{2\pi k_{AF} k}\frac{1}{(2\zeta(3/2)^2)^{1/3}}. \tag{32}$$

The number of particles below the critical temperature, or $T < T_c$ is

$$N = N_1 = \frac{V}{\lambda^3}\frac{\sqrt{2}}{2}\zeta(3/2). \tag{33}$$

For $T > T_c$ we have

$$N = N_1 = \frac{V}{\lambda^3}\frac{\sqrt{2}}{2}g_{3/2}(e^{\beta(\mu-k_{AF})}), \tag{34}$$

the remaining particles are in ground state, so we can write

$$\frac{N}{N_0} = \frac{N-N_1}{N} = 1-\left(\frac{T}{T_c}\right)^{3/2}. \tag{35}$$

The internal energy U for $T < T_c$ is

$$U = N_1(kT)\left[\frac{k_{AF}}{kT} + \frac{3}{2}\frac{g_{5/2}(1)}{g_{3/2}(1)}\right]. \tag{36}$$

Using the Eq. (35) the specific heat is $C_V = \frac{15}{4}Nk\frac{\zeta(5/2)}{\zeta(3/2)}\left(\frac{T}{T_c}\right)^{3/2}$. Above the critical temperature, $T > T_c$, the internal energy U assumes

$$U = N(kT)\left[\frac{3}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)}\right], \tag{37}$$

where z is now set as $z = e^{\beta(\mu-k_{AF})}$. The specific heat reads

$$C_V = \frac{3}{2}Nk\left[\frac{5}{2}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{3}{2}\frac{g_{3/2}(z)}{g_{1/2}(z)}\right]. \tag{38}$$

3.2. The small Lorentz violation Bose gas: $|\vec{p}| \gg k_{AF}$

In this section, the ultrarelativistic ideal Bose gas is studied in SLV, starting from the dispersion relation Carrol–Field–Jackiw (4) in the Bose–Einstein distribution (28). We shall see that the background modifies the thermodynamics of the gas, by introducing a phase transition that does not exist in the standard Bose gas. This effect is very interesting and this result could be important for astrophysical and cosmological applications [22,26–28].

The grand potential is

$$\phi = -kT\,\frac{2\pi V}{h^3}\frac{\beta}{3}\int_0^\pi\int_0^\infty \sin\theta\, p^3\,\frac{ze^{-\beta\left[p+\frac{1}{2}|\cos\theta|\right]}}{1-ze^{-\beta\left[p+\frac{1}{2}|\cos\theta|k_{AF}\right]}}\,dp\,d\theta. \tag{39}$$

Performing the integration over θ and p we obtain

$$\phi = -kT\,\frac{8\pi V}{h^3}\,\frac{1}{\beta^3}\left[\frac{2}{\beta k_{AF}}\left(g_5(\mathrm{e}^{-\beta\mu}) - g_5\left(\mathrm{e}^{-\beta\left(\mu-\frac{k_{AF}}{2}\right)}\right)\right)\right]. \tag{40}$$

In the limit $k_{AF} \longrightarrow 0$ and $\mu = 0$, we have the standard photon gas, the function $g_5(\mathrm{e}^{-\frac{\beta k_{AF}}{2}}) \longrightarrow \zeta(5)$, and the grand potential for the ultrarelativistic Bose–Einstein gas, in the limit where the LSB background does not exist, becomes the well-known result

$$\phi = -kT\,\frac{8\pi V}{h^3}\,\frac{1}{\beta^3} g_4(1), \tag{41}$$

with $g_4(1) = \frac{\pi^4}{90}$.

For further calculations we will write now the grand potential as

$$\phi = -kT\,\frac{8\pi V}{h^3}\,\frac{1}{\beta^3} F_5(z, k_{AF}), \tag{42}$$

where the function $F_5(z, k_{AF})$ will be definite by

$$F_5(z, k_{AF}) = \frac{2}{\beta k_{AF}}\left(g_5(z) - g_5\left(z\mathrm{e}^{-\beta\frac{k_{AF}}{2}}\right)\right) \tag{43}$$

and it is easy to show that its derivative with respect to β is given by

$$\frac{\partial F_5(z, k_{AF})}{\partial\beta} = \frac{1}{\beta}\left(g_4\left(z\mathrm{e}^{-\beta\frac{k_{AF}}{2}}\right) - F_5(z, k_{AF})\right). \tag{44}$$

The thermodynamical quantities are given by

$$P = \frac{8\pi}{h^3}(kT)^4 F_5(z, k_{AF}). \tag{45}$$

When $k_{AF} \longrightarrow 0$ we obtain the well-known result for ultrarelativistic Bose–Einstein gas $P = \frac{8\pi}{h^3}(kT)^4\frac{\pi^4}{90}$.

The internal energy can be calculated by

$$U = \frac{8\pi V}{h^3}(KT)^4\left[4F_5(z, k_{AF}) - g_4\left(z\mathrm{e}^{-\beta\frac{k_{AF}}{2}}\right)\right], \tag{46}$$

the state equation is then

$$\frac{U}{V} = \frac{4F_5(z, k_{AF}) - g_4\left(z\mathrm{e}^{-\beta\frac{k_{AF}}{2}}\right)}{F_5(z, k_{AF})} P. \tag{47}$$

When $k_{AF} \longrightarrow 0$ the energy density is

$$\frac{U}{V} = 3P. \tag{48}$$

In the above equations we can observe that the pressure and the energy density depend only on the temperature and it vanishes as T^4 for $T \longrightarrow 0$. In PV diagrams the isotherms are parallel lines to the V-axis.

3.3. The mean particle number and the Bose temperature

A very useful quantity is the mean particle number, that in Bose–Einstein statistics is given by $N(T, V, z) = \sum_k \frac{1}{z^{-1}\mathrm{e}^{\beta\epsilon_k}-1}$. In the thermodynamic limit we have

$$N = \frac{2\pi V}{h^3}\int_0^{\pi}\int_0^{\infty}\sin\theta\, p^2\,\frac{z\mathrm{e}^{-\beta\left[p+\frac{1}{2}|\cos\theta| k_{AF}\right]}}{1 - z\mathrm{e}^{-\beta\left[p+\frac{1}{2}|\cos\theta| k_{AF}\right]}}\,\mathrm{d}p\,\mathrm{d}\theta \tag{49}$$

or explicitly

$$N = \frac{8\pi V}{h^3}\,\frac{1}{\beta^3}\,\frac{2}{\beta k_{AF}}\left(g_4(k_{AF}) - g_4\left(z\mathrm{e}^{-\beta\frac{k_{AF}}{2}}\right)\right). \tag{50}$$

The maximum mean number of particle is defined when $\mu = \frac{k_{AF}}{2}$ or when we reach the critical temperature T_c. In that temperature we have

$$N = \frac{8\pi V}{h^3}\frac{1}{\beta^3}\frac{2}{\beta k_{AF}}\left(g_4\left(e^{-\beta\frac{k_{AF}}{2}}\right) - \zeta(4)\right). \tag{51}$$

Expanding the above equation for $\frac{k_{AF}}{2kT} \ll 1$ and taking only terms in first order of k_{AF} we have

$$N = \frac{8\pi V}{h^3}(kT)^3\left(\zeta(3) + \frac{\pi^2}{12}\frac{k_{AF}}{2kT}\right). \tag{52}$$

The critical temperature T_c, where the Bose–Einstein condensation occurs is the real cube root of the Eq. (52). The necessary condition for T_c to be real is

$$k_{AF} \leq \frac{18(16\pi^2\zeta(3)^2)^{1/3}}{\pi^2}\frac{\hbar}{c}\left(\frac{N}{V}\right)^{1/3}, \tag{53}$$

where N/V is the density of the condensate. The maximum mass of k_{AF} so that the gas is a condensate in MeV/c^2 is

$$k_{AF} \leq 0.049465\,\rho^{1/4}, \tag{54}$$

where ρ is now the density of the gas in kg/m^3.

When $T = Tc$ the energy of the gas

$$U = NkT\left(\frac{4F_5(z,k_{AF}) - g_4\left(ze^{-\beta\frac{k_{AF}}{2}}\right)}{F_4(z,k_{AF})}\right) \tag{55}$$

becomes

$$U = N_1 kT\left\{\frac{4\left(g_5\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(5)\right) - \frac{\beta k_{AF}}{2}\zeta(4)}{g_4\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(4)}\right\} \tag{56}$$

and the specific heat is given by

$$\begin{aligned} C_V \;=\; & 4Nk\left(\frac{T}{T_c}\right)^3\left\{\frac{4\left(g_5\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(5)\right) - \frac{\beta k_{AF}}{2}\zeta(4)}{g_4\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(4)}\right\} \\ & + (kT)\left(\frac{T}{T_c}\right)^3\frac{\partial}{\partial T}\left\{\frac{4\left(g_5\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(5)\right) - \frac{\beta k_{AF}}{2}\zeta(4)}{g_4\left(e^{\beta\frac{k_{AF}}{2}}\right) - \zeta(4)}\right\}. \end{aligned} \tag{57}$$

The standard case (photon gas and absence of background field) is described in the limit $k_{AF} \to 0$.

$$\frac{C_V}{Nk} = \frac{3\zeta(4)}{\zeta(3)} \approx 2.70118. \tag{58}$$

3.4. *The BEC of an ultrarelativistic Bose gas in SLV with a conserved quantum number*

The behavior of the Bose–Einstein condensate at high temperatures when we have an ideal gas with a conserved quantum number, or generically referred as to "charge", is quite different from the one studied so far. Haber and Weldon [29], for the first time, showed the effects on the critical temperature when antiparticles are introduced in the theory.

The net charge Q of the Bose gas is given by the expression

$$Q = V\sum_k\left[\frac{1}{e^{\beta(\epsilon_k-\mu)}} - \frac{1}{e^{\beta(\epsilon_k+\mu)}}\right]. \tag{59}$$

In our case, by using the dispersion relation given in Eq. (4) and taking the thermodynamic limit in Eq. (59), we find

$$\rho = \frac{N}{V} = \frac{2\pi}{h^3}\int_0^{\pi}\int_0^{\infty}\sin\theta\,p^2\left(\frac{1}{e^{\beta\left[p+\frac{1}{2}|\cos\theta|k_{AF}-\mu\right]}-1} - \frac{1}{e^{\beta\left[p+\frac{1}{2}|\cos\theta|k_{AF}+\mu\right]}-1}\right)dp\,d\theta. \tag{60}$$

After integrating over p and θ, we obtain the explicit result in terms of poly-logarithm functions

$$\rho = \frac{8\pi}{h^3}(kT)^3\frac{2}{\beta k_{AF}}\left[g_4(e^{\beta\mu}) - g_4(e^{-\beta\mu}) - g_4(e^{\beta(\mu-k_{AF}/2)}) + g_4(e^{-\beta(\mu+k_{AF}/2)})\right]. \quad (61)$$

In the regime of high temperatures, when the pair creation is very favorable, $\frac{k_{AF}}{2kT} \ll 1$. The BEC occurs when $\mu = \frac{k_{AF}}{2}$, and net density of charge for the critical temperature T_c reads

$$T_c = \sqrt{\frac{h^3}{4\pi^3k^2}\frac{3|\rho|}{k_{AF}}}. \quad (62)$$

This result is similar to the one obtained for the authors [29] in their seminal work, two decades ago.

4. Final comments

In this paper, we have analyzed how the Lorentz Symmetry breaking background modifies the statistical behavior of a many-particle system by adopting the Carrol–Field–Jackiw dispersion relation. We have computed particles in the classical and quantum regimes, and special attention was devoted to Bose–Einstein statistics in LLV and SLV approach. It was pointed out that in the LLV regime our approach gives rise to the same results obtained by Colladay and McDonald [23], where the thermodynamics quantities must be corrected by the presence of $(k_{AF})^{\mu}$. These results open up the possibilities in bounds on the LSB parameters.

An interesting scenario is the ultrarelativistic thermodynamic approach, where $(k_{AF})^{\mu}$ in SLV condition induces a phase transition on bosonic system. In the paper of Ref. [22], it is argued that the radiation behavior of hot dark matter is close related to a relativistic Bose–Einstein phase transition of a scalar charged field when coupled to gravity, even though this relation is not very well understood [28]. The similarities in the behavior of SFDM BEC and the UltraRelativistic Bose gas under a LSB background could be pointed out as the LSB parameter that is responsible to the phase transition is also very small. In our case, the origin of this term is completely different: it is not a scalar field, but a vector field. It should be pointed out that the masses of these particles are related to the medium, in the same sense as in condensed matter. The background renormalizes the photon mass, and it works like an effective mass.

The results in K_{AF} in our approach point to the novelty of investigating BEC in a different scenario. A very interesting one is the case where we have a BEC in a regime with Large Lorentz Violation and low momenta, induced by K_{AF}. From the literature, we know that the behavior of dark matter is probably closely related to a BEC phase transition of a scalar field in the Planck era. In our case, we have non-scalar particles with mass constrained to the photon mass, which could be a candidate to describe the behavior not only of dark matter but also radiation. In the case of a small Lorentz Violation, our result can be used in the rescaling analysis to interpret K_{AF} in a different era of the universe. This is under investigation [30].

It is interesting to notice that the critical value of the energy density of condensate is related to the LSB parameter by $k_{AF} \leq 0.049465\,\rho^{1/4}$. This could be used for the different universe radiation eras to fix the LSB parameter taking into account the red-shift of the quantities due to the big-bang expansion.

Another important result regards Eq. (62), which relates the critical temperature to the LSB parameter. This could be used to calculate the density of energy and pressure of different species of particles.

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